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A Realization of Riemannian Symmetric Spaces

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§0. Introduction

The purpose of this paper is to construct an imbedding of every Riemannian symmetric space G/K of non-compact type into a compact real analytic manifold $\tilde{X}$. Here G is a semi-simple Lie group and K a maximal compact subgroup. Our imbedding has the following properties:

The action of G on $\tilde{X}$ is analytic and the orbital decomposition of $\tilde{X}$ is of normal crossing type in the sense of Remark 6 in §2. Moreover, there appears the Martin boundary in $\tilde{X}$ and the system of invariant differential equations on the symmetric space has regular singularity along the Martin boundary in the sense of Definition 5.1 in [9].

As for realizations of G/K there are several papers [1], [2], [5], [7], [12], [13], [15] and [4], [10], [11], [14]. If the rank of the symmetric space is higher than one, the Martin boundary does not appear in the realizations given by [1], [2], [5], [7], [12], [13], [15] and the orbital decompositions have more complicated geometrical structures than ours. The realizations given by [4], [10], [11], [14] are essentially the same ones called Satake-Furstenberg compactifications. They are only different in the methods of costructions. There exists a realization among Satake-Furstenberg compactifications where the Martin boundary appears. But it is a compactification of G/K as a

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manifold with boundaries and the natural analytic structure around the boundaries is not investigated. In [8] we construct an imbedding $\tilde{X}'$ of G/K to solve S. Helgason's conjecture by using a result in [9]. But it is not sufficient for further investigations because there is only a local action of G on $\tilde{X}'$. This is a motivation to write this paper. The relation between $\tilde{X}'$ and $\tilde{X}$ is shown in Proposition 11, which says that $\tilde{X}'$ is an open dense submanifold of $\tilde{X}$.

§1. Notation and preliminaries concerning semi-simple Lie groups

We will use the standard notation $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the ring of integers, the field of real numbers and the field of complex numbers, respectively. The set of non-negative integers is denoted by $\mathbb{N}$ and the set of positive real numbers by $\mathbb{R}_+$. Lie groups will be denoted by Latin capital letters and their Lie algebras by corresponding small German letters. If C is a Lie group and $\mathfrak{c}$ its Lie algebra, the adjoint representation of C is denoted by Ad (or Ad_C) and the adjoint representation of $\mathfrak{c}$ by ad (or $\mathrm{ad}_{\mathfrak{c}}$).

We will now list some standard notation concerning semi-simple Lie groups used in this paper and subsequent papers. Let G be a connected semi-simple Lie group with finite center Z, $\mathfrak{g}$ the Lie algebra of G and $\langle\ ,\ \rangle$ the <u>Killing form</u> of $\mathfrak{g}$. Let θ be a <u>Cartan involution</u> of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the <u>Cartan decomposition</u> of $\mathfrak{g}$ into the eigenspaces of θ. We also denote by θ the Cartan involution of G corresponding to the Cartan involution θ of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{a}^*$ its dual, $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$. If $\lambda, \mu \in \mathfrak{a}^*_{\mathbb{C}}$, let $H_\lambda \in \mathfrak{a}_{\mathbb{C}}$ be determined by $\lambda(H) = \langle H_\lambda , H \rangle$ for $H \in \mathfrak{a}$ and put $\langle \lambda , \mu \rangle = \langle H_\lambda , H_\mu \rangle$. Let $\mathfrak{j}$ be a <u>Cartan subalgebra</u> of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{j} = \mathfrak{a} + \mathfrak{t}$ where $\mathfrak{t} = \mathfrak{j} \cap \mathfrak{k}$. We denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and for any subspace $\mathfrak{b}$ of $\mathfrak{g}$ we denote by $\mathfrak{b}_{\mathbb{C}}$ the complex linear subspace of $\mathfrak{g}_{\mathbb{C}}$ spanned by $\mathfrak{b}$. For any <u>root</u> α of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, we fix a root vector X_α corresponding to α. Introducing compatible orders in the space of real valued linear forms on $\mathfrak{a} + \sqrt{-1}\,\mathfrak{t}$ and $\mathfrak{a}$, we denote by P_+ the set of non-zero positive roots α such that $\alpha|_{\mathfrak{a}} \neq 0$, by Σ the set of restricted roots, by Σ^+ the set of restricted positive roots and by $\Psi = \{\alpha_1, \ldots, \alpha_\ell\}$ the set of restricted positive simple roots. Let ρ denote half the sum of the positive restricted roots with multiplicity, that is, $2\rho = (\sum_{\alpha \in P_+} \alpha)|_{\mathfrak{a}_{\mathbb{C}}}$. For any root α in Σ, we denote by $\mathfrak{g}^\alpha$ the root space in $\mathfrak{g}$ corresponding to α. We put $\mathfrak{n}^+ = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$ and $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$, then $\mathfrak{n}^+ = \mathfrak{g} \cap \sum_{\alpha \in P_+} \mathbb{C}X_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha \in \Sigma^-} \mathfrak{g}^\alpha$, where Σ^- denotes the set of negatives of the members in Σ^+. Let K, A, N^+ and N^- denote the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively. Let M denote the centralizer of A in K, M* the normalizer of A in K and W the factor group M*/M, the (little) Weyl group. The Weyl group W acts as a group of linear transformations of $\mathfrak{a}$ and also on $\mathfrak{a}^*_{\mathbb{C}}$ by $(w\lambda)(H) = \lambda(w^{-1}H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $w \in W$. For any element w in W, we fix its representative m_w in M*. We put $\mathfrak{a}^+ = \{H \in \mathfrak{a} \,;\, \alpha(H) > 0$ for any α in $\Sigma^+\}$, which is called the positive Weyl chamber. Let $A^+ = \exp \mathfrak{a}^+$, $A' = \bigcup_{w \in W} \mathrm{Ad}(m_w)A^+$ and $P = MAN$. Then A' is the totality of regular elements in A, P is a minimal parabolic subgroup of G and there exist the decompositions

(1.1) $\quad G = K\overline{A^+}K \quad$ (Cartan decomposition),

(1.2) $\quad G = KAN^+ \quad$ (Iwasawa decomposition),

(1.3) $\quad G = \bigcup_{w \in W} Pm_wP \quad$ (Bruhat decomposition).

Here $\overline{A^+}$ is the closure of A^+ in G and in (1.2) each $g \in G$ can be uniquely written

(1.4) $\quad g = k(g)\exp H(g)\, n(g), \quad k(g) \in K,\ H(g) \in \mathfrak{a},\ n(g) \in N^+.$

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, which is naturaly identified with $\mathbb{D}(G)$, the totality of the left

G-invariant differential operators on G. The number ℓ which equals dim $\mathfrak{a}$ is called the real rank of G and the rank of the symmetric space G/K. Let $\mathbb{D}(G/K)$ denote the algebra of left G-invariant differential operators on G/K and put $\mathbb{D}(G)^K = \{D \in U(\mathfrak{g});\ \mathrm{Ad}(k)D = D$ for any $k \in K\}$. Then $\mathbb{D}(G/K)$ is a polynomial ring over $\mathbb{C}$ with ℓ algebraically independent generators and there exists a natural homomorphism of $\mathbb{D}(G)^K$ onto $\mathbb{D}(G/K)$.

For an element w in W, we define subalgebras $\mathfrak{n}_w^+$, $\mathfrak{u}_w^+$ and $\mathfrak{u}_w^-$ of $\mathfrak{g}$ by

$$(1.5)\qquad \begin{aligned} &\mathfrak{n}_w^+ = \mathfrak{n}^+ \cap \mathrm{Ad}(m_w)\mathfrak{n}^+, \quad \mathfrak{u}_w^+ = \mathfrak{n}^+ \cap \mathrm{Ad}(m_w)\mathfrak{n}^- \\ &\mathfrak{u}_w^- = \mathrm{Ad}(m_w^{-1})\mathfrak{u}_w^+ = \mathfrak{n}^- \cap \mathrm{Ad}(m_w^{-1})\mathfrak{n}^+. \end{aligned}$$

We put $N_w^+ = \exp(\mathfrak{n}_w^+)$, $U_w^+ = \exp(\mathfrak{u}_w^+)$ and $U_w^- = \exp(\mathfrak{u}_w^-)$, then they are closed simply connected subgroups of G and

$$(1.6)\qquad N^+ = N_w^+U_w^+ = U_w^+N_w^+, \quad N_w^+ \cap U_w^+ = \{1\}.$$

The Killing form defines a Euclidian inner product on $\mathfrak{a}^*$ and $\alpha_i \in \Psi$ $(i = 1,\ldots,\ell)$ defines the reflection $w_{\alpha_i}: \lambda \longmapsto \lambda - 2\alpha_i\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$ on $\mathfrak{a}^*$. We can naturally identify W with the reflection group generated by $w_{\alpha_1},\ldots,w_{\alpha_\ell}$. Let $w = w_1\cdots w_n$ is the minimal expression for $w \in W$ as a product of reflections with respect to the roots in Ψ, then the length $L(w)$ of w is said to be n. Let Θ be the subset of Ψ and W_Θ be the subgroup of W generated by the reflections with respect to the elements in Θ. We note here that the number of the subsets of Ψ equals 2^ℓ. We put

$$(1.7)\qquad \begin{aligned} &\langle\Theta\rangle^+ = \Sigma^+ \cap \sum_{\alpha_i\in\Theta} \mathbb{R}\alpha_i, \\ &W(\Theta) = \{w \in W;\ w^{-1}\langle\Theta\rangle^+ \subset \Sigma^+\}. \end{aligned}$$

Then every element w in W can be written in one and only one way in the form (cf. Proposition 1.1.2.13 in [16])

(1.8) $\qquad w = w_{\Theta}\, w(\Theta),\quad w_{\Theta} \in W_{\Theta}, w(\Theta) \in W(\Theta).$

Let w^* denote the unique element in W such that $\mathrm{Ad}(w^*)\,\mathfrak{n}^+ = \mathfrak{n}^-$. Then $L(w^*) \geqq L(w)$ for any $w \in W$ and $L(w^*) = L(w)$ means $w = w^*$. Let w^*_{Θ} and $w^*(\Theta)$ denote the elements in W_{Θ} and $W(\Theta)$, respectively, such that $w^* = w^*_{\Theta}\, w^*(\Theta)$. Put $P_{\Theta} = \bigcup_{w \in W_{\Theta}} P\, m_w P$. Then P_{Θ} constitute the <u>parabolic subgroups</u> containing P when Θ runs through the subsets of Ψ. We define subalgebras $\mathfrak{a}_{\Theta}$, $\mathfrak{a}(\Theta)$, $\mathfrak{n}^{\pm}_{\Theta}$, $\mathfrak{n}^{\pm}(\Theta)$, $\mathfrak{m}_{\Theta}$ and $\mathfrak{m}_{\Theta}(K)$ of $\mathfrak{g}$ by

$$\mathfrak{a}_{\Theta} = \{ H \in \mathfrak{a} \,;\ \alpha(H) = 0 \text{ for every } \alpha \text{ in } \Theta \},$$

$$\mathfrak{a}(\Theta) = \{ H \in \mathfrak{a} \,;\ \langle H, X \rangle = 0 \text{ for any } X \text{ in } \mathfrak{a}_{\Theta} \},$$

(1.9)
$$\mathfrak{n}^+_{\Theta} = \sum_{\lambda \in \Sigma^+ - \langle \Theta \rangle^+} \mathfrak{g}^{\lambda}, \quad \mathfrak{n}^-_{\Theta} = \theta(\mathfrak{n}^+_{\Theta}),$$

$$\mathfrak{n}^+(\Theta) = \sum_{\lambda \in \langle \Theta \rangle^+} \mathfrak{g}^{\lambda}, \quad \mathfrak{n}^-(\Theta) = \theta(\mathfrak{n}^+(\Theta)),$$

$$\mathfrak{m}_{\Theta} = \mathfrak{m} + \mathfrak{n}^+(\Theta) + \mathfrak{n}^-(\Theta) + \mathfrak{a}(\Theta),$$

$$\mathfrak{m}_{\Theta}(K) = \mathfrak{m}_{\Theta} \cap \mathfrak{k}.$$

Let A_{Θ} , $A(\Theta)$, $N^{\pm}_{\Theta}$, $N^{\pm}(\Theta)$, $M_{\Theta,0}$ and $M_{\Theta}(K)_0$ denote the connected analytic subgroups of G corresponding, respectively, to $\mathfrak{a}_{\Theta}$, $\mathfrak{a}(\Theta)$, $\mathfrak{n}^{\pm}_{\Theta}$, $\mathfrak{n}^{\pm}(\Theta)$, $\mathfrak{m}_{\Theta}$ and $\mathfrak{m}_{\Theta}(K)$. Then $A_{\Theta}\, N^{\pm}_{\Theta}$ is a closed solvable subgroups of G and we have the direct decomposition

(1.10) $$A = A_{\Theta}\, A(\Theta)$$

and the semi-direct decomposition

(1.12) $$N^{\pm} = N^{\pm}_{\Theta}\, N^{\pm}(\Theta).$$

We put $M_{\Theta} = M\, M_{\Theta,0}$ and $M_{\Theta}(K) = M\, M_{\Theta}(K)_0$, then the group $M_{\Theta} A_{\Theta}$ is the centralizer of $\mathfrak{a}_{\Theta}$ in G, $M_{\Theta}(K) = K \cap M_{\Theta}$ and we have the decompositions (cf. §1.2.4 in [16])

(1.12) $\quad M_{\Theta} = M_{\Theta}(K)\, A(\Theta)\, N^{\pm}(\Theta) \qquad$ (Iwasawa decomposition),

(1.13) $\quad P_{\Theta} = M_{\Theta}\, A_{\Theta}\, N^+_{\Theta} \qquad$ (Langlands decomposition),

(1.14) $\quad P_{\Theta} = M_{\Theta}(K)\, A\, N^+,$

(1.15) $\quad G = \bigcup_{w \in W(\Theta)^{-1}} N^{+} m_w P_{\Theta}$ (disjoint union),

(1.16) $\quad G = \bigcup_{w \in W(\Theta)^{-1}} U_w^{+} m_w P_{\Theta}$ (disjoint union).

The decompositions (1.12), (1.13) and (1.14) give analytic diffeomorphisms of the product manifolds $M_{\Theta}(K) \times A(\Theta) \times N^{\pm}(\Theta)$, $M_{\Theta} \times A_{\Theta} \times N_{\Theta}^{+}$, $M_{\Theta}(K) \times A \times N^{+}$ onto M_{Θ}, P_{Θ} and P_{Θ}, respectively, and if w is in W_{Θ}, the map $(u,p) \longmapsto u\, m_w\, p$ defines an analytic diffeomorphism of the product manifold $U_w^{+} \times P_{\Theta}$ onto the submanifold $N^{+} m_w P_{\Theta}$ of G. Here we note

(1.17) $\quad Ad(m_w^{-1})\, U_w^{+} \subset N_{\Theta}^{-} = Ad(m_{w*(\Theta)})\, U_{w*(\Theta)^{-1}}^{+} \quad$ for $w \in W(\Theta)$.

Hence G is the union of the open submanifold $N_{\Theta}^{-} P_{\Theta}$ and submanifolds of lower dimensions.

§2. A realization of symmetric spaces in compact manifolds

In this section we will construct a compact manifold $\widetilde{X}$ such that G acts analytically on G and that the open G-orbits are isomorphic to symmetric spaces. To investigate all the G-orbits appeared in $\widetilde{X}$, we prepare the following lemma.

<u>Lemma 1.</u> Put $P_\Theta(K) = M_\Theta(K)\, A_\Theta\, N_\Theta^+$. Then $P_\Theta(K)$ is a closed subgroup of G and there exist the decompositions

(2.1) $G = \bigcup_{w \in W(\Theta)^{-1}} m_w U_w^- N^-(\Theta)\, A(\Theta)\, P_\Theta(K)$ (disjoint union),

(2.2) $G = \bigcup_{w \in W} m_w N^- A(\Theta)\, P_\Theta(K)$.

If $w \in W(\Theta)^{-1}$, the map $(u_w, n, a, p) \longmapsto u_w nap$ defines an analytic diffeomorphism of the product manifold $U_w^- \times N^-(\Theta) \times A(\Theta) \times P_\Theta(K)$ onto the submanifold $U_w^- N^-(\Theta)\, A(\Theta)\, P_\Theta(K)$ in G. And G is a union of the open dense submanifold $N^- A(\Theta)\, P_\Theta(K)$ and submanifolds of lower dimensions.

<u>Proof.</u> To show $P_\Theta(K)$ is a group we need only verify $ma = am$, $aN_\Theta^+ a^{-1} \subset N_\Theta^+$ and $\mathrm{Ad}(m) N_\Theta^+ \subset N_\Theta^+$ for $m \in M_\Theta(K)$ and $a \in A_\Theta$. But they clearly follow from (1.9) and the definition of $M_\Theta(K)$. The groups $M_\Theta(K)$, A_Θ and N_Θ^+ are closed in K, A and N^+, respectively. Therefore $P_\Theta(K)$ is closed in G because of the Iwasawa decomposition (1.2). Next we note that $M_\Theta = N^-(\Theta)\, A(\Theta)\, M_\Theta(K)$ (cf. (1.12)). Then (1.5), (1.13) and (1.12) imply that in (1.16)

$$\begin{aligned} U_w^+ m_w P_\Theta &= m_w (N^- \cap m_w^{-1} N^+ m_w) N^-(\Theta)\, A(\Theta)\, M_\Theta(K)\, A_\Theta\, N_\Theta^+ \\ &= m_w U_w^- N^-(\Theta)\, A(\Theta)\, P_\Theta(K) \end{aligned}$$

for $m_w \in W(\Theta)^{-1}$ and that

$$\begin{aligned} N_\Theta^- P_\Theta &= N_\Theta^- N^-(\Theta)\, A(\Theta)\, M_\Theta(K)\, A_\Theta\, N_\Theta^+ \\ &= N^- A(\Theta)\, P_\Theta(K). \end{aligned}$$

This proves the rest part of Lemma 1. q.e.d.

Remark 2. Suppose $\Theta = \Psi$. Then $W_\Theta = W$, $W(\Theta) = \{1\}$, $M_\Theta = G$, $M_\Theta(K) = G \cap K = K$, $A_\Theta = \{1\}$, $A(\Theta) = A$, $N^\pm_\Theta = 1$, $N^\pm(\Theta) = N^\pm$, $P_\Theta = G$, $P_\Theta(K) = K$ and (2.1) is reduced to $G = N^- A K$ (Iwasawa decomposition). On the other hand, suppose $\Theta = \phi$. Then $W_\Theta = \{1\}$, $W(\Theta) = W$, $M_\Theta = M$, $M_\Theta(K) = M$, $A_\Theta = A$, $A(\Theta) = \{1\}$, $N^\pm_\Theta = N^\pm$, $N^\pm(\Theta) = \{1\}$, $P_\Theta = P_\Theta(K) = MAN^+ = P$ and (2.2) equals $G = \bigcup_{w \in W} m_w N^- P$.

If C is a Lie group and $\mathfrak{c}$ is its Lie algebra, we identify $\mathfrak{c}$ with the totality of left invariant vector fields on C. Fix a basis $\{Y_1,\ldots,Y_m\}$ of $\mathfrak{c}$. Then any real analytic vector field Y on C can be uniquely expressed as

$$Y = \sum_{i=1}^{m} c_i(p) Y_i$$

with real analytic functions $c_i(p)$ on C. This is clear because for any point p in C, $\{(Y_1)_p,\ldots,(Y_m)_p\}$ is a basis of the tangent space T_pC of C at p. Let $\{H_1,\ldots,H_\ell\}$ be the dual basis of $\mathfrak{a}$ with respect to $\Psi = \{\alpha_1,\ldots,\alpha_\ell\}$, that is, $\alpha_i(H_j) = \delta_{ij}$. For $\lambda \in \Sigma^+$, we fix a basis $\{X_{\lambda_i};\ 1 \le i \le m(\lambda)\}$ of $\mathfrak{g}^\lambda$, where $m(\lambda) = \dim \mathfrak{g}^\lambda$ and put $X_{-\lambda_i} = -\theta(X_{\lambda_i})$.

Lemma 3. Let $\tilde{X}_\Theta$ be the homogeneous space $G/P_\Theta(K)$. Fix an element g in G and identify $N^- \times A(\Theta)$ with the open dense submanifold of $\tilde{X}_\Theta$ by the map $(n,a) \longmapsto gnaP_\Theta(K)$ (cf. Lemma 1). For an element Y in $\mathfrak{g}$, let $Y|\tilde{X}_\Theta$ be the vector field on $\tilde{X}_\Theta$ corresponding to the 1-parameter group which is defined by the action $\exp(tY)$ on $\tilde{X}_\Theta$ $(t \in \mathbb{R})$. Then at any point $p = (n,a)$ in

$N^- \times A(\Theta)$, the vector field is expressed as

$$
\begin{aligned}
(Y|\tilde{X}_{\Theta})_p = & \textstyle\sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda_i}(g,n)(X_{-\lambda_i})_p \\
& + \textstyle\sum_{\lambda \in \langle\Theta\rangle^+} \sum_{i=1}^{m(\lambda)} c_{\lambda_i}(g,n) e^{-2\lambda \log a} (X_{-\lambda_i})_p \qquad (2.3) \\
& + \textstyle\sum_{\alpha_i \in \Theta} c_i(g,n)(H_i)_p
\end{aligned}
$$

by the identification $T_n N^- \oplus T_a A(\Theta) \simeq T_p(N^- \times A(\Theta)) \simeq T_{gnaP_\Theta(K)} \tilde{X}_\Theta$. Here the real analytic functions $c_{\pm\lambda_i}(g,n)$ and $c_i(g,n)$ are determined by the equation

$$
\begin{aligned}
\mathrm{Ad}^{-1}(gn)Y = & \textstyle\sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda_i}(g,n)X_{\lambda_i} + c_{-\lambda_i}(g,n) \\
& X_{-\lambda_i}) + \textstyle\sum_{i=1}^{\ell} c_i(g,n)H_i + M(g,n), \quad M(g,n) \in \mathfrak{m}. \qquad (2.4)
\end{aligned}
$$

<u>Proof.</u> Assume $|t|$ is sufficiently small. Then the direct sum decompositions $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{a} + \mathfrak{n}^+ + \mathfrak{m} = \mathfrak{n}^- + \mathfrak{a}(\Theta) + \mathfrak{m}_\Theta(K) + \mathfrak{a}_\Theta + \mathfrak{n}_\Theta^+$ and the relation $[\mathfrak{a}, \mathfrak{n}^-] \subset \mathfrak{n}^-$ show that we can put

$$
\begin{aligned}
& \exp(tY)\, gn = gn \exp N_1^-(t) \exp A_1(t) \exp N_1^+(t) \exp M_1(t), \\
(2.5) \quad & \exp N_1^+(t)\, a = a \exp N_2^-(t) \exp A_2(t) \exp P_2(t), \\
& \exp N_1^-(t) \exp A_1(t)\, a \exp N_2^-(t)\, a^{-1} = \exp N_3^-(t) \exp A_1(t),
\end{aligned}
$$

where $N_i^-(t) \in \mathfrak{n}^-$ $(i=1,2,3)$, $N_1^+(t) \in \mathfrak{n}^+$, $A_1(t) \in \mathfrak{a}$, $A_2(t) \in \mathfrak{a}(\Theta)$, $M_1(t) \in \mathfrak{m}$ and $P_2(t) \in \mathfrak{m}_\Theta(K) + \mathfrak{a}_\Theta + \mathfrak{n}_\Theta^+$. Hence we have

$$
(2.6) \quad \exp(tY)\, gnaP_\Theta(K) = gn \exp N_3^-(t)\, a \exp(A_1(t) + A_2(t))\, P_\Theta(K).
$$

Put $(\partial N_i^-(t)/\partial t)(0) = N_i^-$ $(i=1,2,3)$, $(\partial N_1^+(t)/\partial t)(0) = N_1^+$ and $(\partial A_j(t)/\partial t)(0) = A_j$ $(j=1,2)$. Then (2.5) shows that

$$
\begin{aligned}
& \mathrm{Ad}^{-1}(gn)Y \equiv N_1^- + A_1 + N_1^+ \quad \mathrm{mod}\ \mathfrak{m}, \\
(2.7) \quad & \mathrm{Ad}^{-1}(a)N_1^+ \equiv N_2^- + A_2 \quad \mathrm{mod}\ \mathfrak{m}_\Theta(K) + \mathfrak{a}_\Theta + \mathfrak{n}_\Theta^+, \\
& N_1^- + \mathrm{Ad}(a)N_2^- = N_3^-.
\end{aligned}
$$

If $\lambda \in \langle\Theta\rangle^+$, we have

$$
\begin{aligned}
\mathrm{Ad}^{-1}(a)X_{\lambda_i} & = e^{-\lambda \log a} X_{\lambda_i} \\
& = e^{-\lambda \log a}(X_{\lambda_i} - X_{-\lambda_i}) + e^{-\lambda \log a} X_{-\lambda_i}
\end{aligned}
$$

$$\equiv e^{-2\lambda \log a} \, Ad^{-1}(a) X_{-\lambda_i} \mod \mathfrak{m}_{\Theta}(K).$$

On the other hand, if $\lambda \in \Sigma^+ - \langle \Theta \rangle^+$, we have

$$Ad^{-1}(a) X_{\lambda_i} = e^{-\lambda \log a} X_{\lambda_i} \in \mathfrak{n}^+_{\Theta}.$$

Then $A_2 = 0$ and

$$N_1^- + Ad^{-1}(a) N_2^- = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda_i}(g,n) X_{-\lambda_i}$$
$$+ \sum_{\lambda \in \langle \Theta \rangle^+} \sum_{i=1}^{m(\lambda)} c_{\lambda_i}(g,n) e^{-2\lambda \log a} X_{-\lambda_i},$$

$$A_1 + A_2 \equiv \sum_{\alpha_i \in \Theta} c_i(g,n) H_i \mod \mathfrak{a}_{\Theta}.$$

Thus we obtain (2.3) by (2.6) and (2.7). q.e.d.

Let $\hat{X}$ be the product manifold $G \times N^- \times \mathbb{R}$ and let $\hat{x} = (g,n,t)$ be a point in $\hat{X}$ ($g \in G$, $n \in N^-$, $t = (t_1, \ldots, t_\ell) \in \mathbb{R}^\ell$). Then G acts on $\hat{X}$ by the correspondence $(g',(g,n,t)) \longmapsto (g'g,n,t)$ for $g' \in G$. Put $\operatorname{sgn} \hat{x} = (\operatorname{sgn} t_1, \ldots, \operatorname{sgn} t_\ell) \in \{-1,0,1\}^\ell$, $\Theta_{\hat{x}} = \{\alpha_i \in \Psi \, ; \, t_i \neq 0\}$ and $a(\hat{x}) = \exp(-\sum_{t_i \neq 0} H_i \log |t_i|) \in A(\Theta_{\hat{x}})$, where $\operatorname{sgn} s = s/|s|$ for $s \in \mathbb{R} - \{0\}$ and $\operatorname{sgn} 0 = 0$. We will define an equivalence relation for points in $\hat{X}$.

<u>Definition 4.</u> Two points $\hat{x} = (g,n,t)$ and $\hat{x}' = (g',n',t')$ in $\hat{X}$ are equivalent, which will be denoted by $\hat{x} \sim \hat{x}'$, if and only if the following two conditions hold.

(2.8) $$\operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}'.$$

(2.9) $$g n a(\hat{x}) P_{\Theta_{\hat{x}}}(K) = g' n' a(\hat{x}') P_{\Theta_{\hat{x}}}(K) \quad \text{in } \tilde{X}_{\Theta_{\hat{x}}}.$$

Then we denote by $\tilde{X}$ the quotient space of $\hat{X}$ with the quotient topology defined by the equivalence relation.

Since the action of G on $\hat{X}$ is compatible with the equivalence relation, G also acts on $\widetilde{X}$. Let π be the natural projection of $\hat{X}$ onto $\widetilde{X}$. Put $\widetilde{U}_g = \pi(\{g\}\times N^-\times \mathbb{R}^l)$ for $g\in G$. Then the map $(n,t) \longmapsto ((g,n,t))$ defines the bijection (cf. Lemma 1)

(2.10) $$\varphi_g : N^-\times\mathbb{R}^l \xrightarrow{\sim} \widetilde{U}_g.$$

Theorem 5. The quotient space $\widetilde{X}$ has the following properties.

i) $\widetilde{X}$ is a simply connected, compact, real analytic manifold without boundary.

ii) $$\widetilde{X} = \bigcup_{w\in W} \widetilde{U}_{m_w} = \bigcup_{g\in G} \widetilde{U}_g.$$

Here $\widetilde{U}_g$ is an open submanifold of $\widetilde{X}$ with the topology such that the map (2.10) is a real analytic diffeomorphism. Moreover $\widetilde{X}-\widetilde{U}_g$ is a union of a finite number of submanifolds of $\widetilde{X}$ whose codimensions in $\widetilde{X}$ are not lower than 2.

iii) The action of G on $\widetilde{X}$ is real analytic and for a point $\hat{x}$ in $\hat{X}$, the G-orbit of $\pi(\hat{x})$ is isomorphic to the homogeneous space $G/P_{\Theta_{\hat{x}}}(K)$ and for points $\hat{x}$ and $\hat{x}'$ in $\hat{X}$, the G-orbits of $\pi(\hat{x})$ and $\pi(\hat{x}')$ are coincide if and only if $\operatorname{sgn}\hat{x} = \operatorname{sgn}\hat{x}'$. Hence the orbital decomposition of $\widetilde{X}$ with respect to the action of G is of the form

$$\widetilde{X} \simeq \bigcup_{\Theta\subset\Psi} 2^{\#\Theta}(G/P_\Theta(K)) \qquad \text{(disjoint union)},$$

where $\#\Theta$ is the number of the elements of Θ and $2^{\#\Theta}(G/P_\Theta(K))$ is the disjoint union of $2^{\#\Theta}$ copies of $G/P_\Theta(K)$.

iv) Identify the open G-orbit $\pi(\{\hat{x}\in\hat{X};\ \operatorname{sgn}\hat{x} = (1,\dots,1)\})$ with the Riemannian symmetric space G/K and the orbit of the lowest dimension $\pi(\{\hat{x}\in\hat{X};\ \operatorname{sgn}\hat{x} = 0\})$ with its Martin boundary G/P. Let $\mathbb{D}(\widetilde{X})$ be the totality of G-invariant differential operators on $\widetilde{X}$ whose coefficients are real analytic functions. Then the natural restriction

$$\mathbb{D}(\tilde{X}) \xrightarrow{\sim} \mathbb{D}(G/K)$$

is bijective. For any homomorphism χ of $\mathbb{D}(\tilde{X})$ to $\mathbb{C}$ as algebras, the system of differential equations on $\tilde{X}$

$$\mathcal{M}_{\chi} : (D - \chi(D))u = 0 \quad \text{for } D \in \mathbb{D}(\tilde{X})$$

has regular singularity along the set of walls $\tilde{X}_i = \pi(\{(g,n,t) \in \hat{X};\ t_i = 0\})$ with the edge G/P in the sense of Definition 5.1 in [9].

<u>Remark 6.</u> Since

$$(2.11) \qquad \dim \tilde{X} - \dim G/P_{\Theta}(K) = \ell - \#\Theta,$$

the open G-orbits in $\tilde{X}$ are isomorphic to G/K and the number of them equals 2^{ℓ} and that of all the G-orbits equals 3^{ℓ}. The decomposition of $\tilde{X}$ into G-orbits is of "normal crossing type" in the following sense:

For every point in $\tilde{X}$, there exists a local coordinate system $(x_1,\ldots,x_k,y_1,\ldots,y_{\ell})$ on a neighbourhood of the point such that if $\operatorname{sgn} y_j = \operatorname{sgn} y'_j$ for $j = 1,\ldots,\ell$, two points $(x_1,\ldots, x_k,y_1,\ldots,y_{\ell})$ and $(x'_1,\ldots,x'_k,y'_1,\ldots,y'_{\ell})$ belong to the same G-orbit.

For example, put $G = SL(2,\mathbb{R})$, $N^- = \left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} ;\ x \in \mathbb{R} \right\}$, $A = \left\{ \begin{pmatrix} 1/\sqrt{t} & \\ & \sqrt{t} \end{pmatrix} ;\ t \in \mathbb{R}_+ \right\}$ and $z = x + \sqrt{-1}\,t$. Then we can easily show that $\tilde{X}$ is isomorphic to the 1-dimensional complex projective space $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the action of G

$$G \times \mathbb{P}^1_{\mathbb{C}} \ni \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \longmapsto \frac{c + dz}{a + bz} \in \mathbb{P}^1_{\mathbb{C}}$$

and that

$$\tilde{X} = U_1 \cup U_{m_{w^*}} \simeq \underset{z}{\mathbb{C}} \cup \underset{-1/z}{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}.$$

For the first step to prove Theorem 5, we prepare

Lemma 7. The map

$$(2.12)\qquad \varphi_{g'}^{-1}\circ\varphi_g : \varphi_g^{-1}(\tilde U_g\cap\tilde U_{g'}) \longrightarrow \varphi_{g'}^{-1}(\tilde U_g\cap\tilde U_{g'})$$

is an analytic diffeomorphism between the open subsets of $N^-\times\mathbb{R}^\ell$.

Proof. Let Y be an element of $\mathfrak{g}$. By the identification

$$\begin{array}{ccccccccc} G/K & \simeq & N^-\times A & \xrightarrow{\sim} & N^-\times\mathbb{R}^\ell_+ & \hookrightarrow & N^-\times\mathbb{R}^\ell & \overset{\varphi_g^{-1}}{\xleftarrow{\sim}} & \tilde U_{g'} \\ \Psi & & \Psi & & \Psi & & & & \\ gnaK & \leftarrow & (n,a) & \mapsto & (n, e^{-\alpha_1 \log a}, \dots, e^{-\alpha_\ell \log a}) = (n,t) & & & & \end{array}$$

the vector field $Y|N^-\times\mathbb{R}^\ell_+$ corresponding to the 1-parameter group defined by the action exp(sY) on G/K for $s\in\mathbb{R}$ is expressed as

$$(2.13)\qquad \begin{aligned} Y|N^-\times\mathbb{R}^\ell_+ = & \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} (c_{\lambda_i}(g,n)t^{2\lambda}+c_{-\lambda_i}(g,n))X_{-\lambda_i} \\ & -\sum_{i=1}^{\ell} c_i(g,n)\,t_i\partial/\partial t_i . \end{aligned}$$

Here we denote by $t^{2\lambda}$ the function $t_1^{2\lambda(H_1)}\cdots t_\ell^{2\lambda(H_\ell)}$ and the functions $c_{\pm\lambda_i}(g,n)$ and $c_i(g,n)$ are those which are determined by (2.4) (see Lemma 3). Since $\lambda(H_i)$ are non-negative integers for $\lambda\in\Sigma^+$, the vector field $Y|N^-\times\mathbb{R}^\ell_+$ is analytically extended to a vector field $Y|N^-\times\mathbb{R}^\ell$ on $N^-\times\mathbb{R}^\ell$.

For every point $\hat x=(g,\hat n,\hat t)$ in $\hat X$, put $B_{\hat x}=\{(t_1,\dots,t_\ell)\in\mathbb{R}^\ell ;$ $\operatorname{sgn} t_i = \operatorname{sgn}\hat t_i$ for $1\le i\le\ell\}$ and define the identification

$$\begin{array}{ccccccccc} G/P_{\Theta_{\hat x}} & \simeq & N^-\times A(\Theta_{\hat x}) & \xrightarrow{\sim} & N^-\times B_{\hat x} & \hookrightarrow & N^-\times\mathbb{R}^\ell & \overset{\varphi_g^{-1}}{\xleftarrow{\sim}} & \tilde U_g . \\ \Psi & & \Psi & & \Psi & & & & \\ gnaP_{\Theta_{\hat x}} & \leftarrow & (n,a) & \mapsto & (n, \operatorname{sgn}\hat t_1\, e^{-\alpha_1\log a}, \dots, \operatorname{sgn}\hat t_\ell\, e^{-\alpha_\ell \log a}) & & & & \end{array}$$

Since (2.13) shows $(Y|N^-\times\mathbb{R}^\ell)_q \in T_q(N^-\times B_{\hat x})$ for $q\in N^-\times B_{\hat x}$, we can restrict the vector field $Y|N^-\times\mathbb{R}^\ell$ on $N^-\times B_{\hat x}$. Then, using the above identification and comparing (2.3) and (2.13), we see that its restriction on $N^-\times B_{\hat x}$ is the same one defined by (2.3).

Hence by Definition 4 we have the following claim:

Suppose $\hat{x} = (g,\hat{n},\hat{t})$ in $\hat{X}$ and Y in $\mathfrak{g}$ satisfying that $g\hat{n}a(\hat{x}) \in \exp(sY)gN^-A(\Theta_{\hat{x}})P_{\Theta_{\hat{x}}}(K)$ for $0 \le s \le 1$. Then there exists an open subset V of $N^- \times \mathbb{R}^\ell$ containing $(\hat{n},\hat{t})$ such that $\varphi^{-1}_{(\exp Y)g} \circ \varphi_g$ defines an analytic diffeomorphism of V to an open subset of $N^- \times \mathbb{R}^\ell$.

For any $\hat{x} = (g,\hat{n},\hat{t})$, there exist $Y_1,\dots,Y_k \in \mathfrak{n}^- + \mathfrak{a}(\Theta_{\hat{x}})$ such that $\hat{n}a(\hat{x}) = \exp Y_k \exp Y_{k-1} \cdots \exp Y_1$. Put $y(s) = \exp\{(s-[s])\mathrm{Ad}(g)Y_{[s]+1}\} \exp(\mathrm{Ad}(g)Y_{[s]}) \cdots \exp(\mathrm{Ad}(g)Y_1)$ for $0 \le s \le k$, where $[s]$ is the largest integer satisfying $[s] \le s$. Then $y(s)gN^-A(\Theta_{\hat{x}})P_{\Theta_{\hat{x}}}(K) = gN^-A(\Theta_{\hat{x}})P_{\Theta_{\hat{x}}}(K)$ and $y(k)g = g\hat{n}a(\hat{x})$. Applying the above claim to $y(s)$ in place of $\exp(sY)$, we see that $\varphi^{-1}_{g\hat{n}a(\hat{x})} \circ \varphi_g$, which equals $(\varphi^{-1}_{y(k)g} \circ \varphi_{y(k-1)g}) \circ \cdots \circ (\varphi^{-1}_{y(2)g} \circ \varphi_{y(1)g}) \circ (\varphi^{-1}_{y(1)g} \circ \varphi_g)$, defines an analytic diffeomorphism of a suitable neighbourhood of $(\hat{n},\hat{t})$ to a neighbourhood of $(1, \mathrm{sgn}\,\hat{x})$.

Let $\tilde{q}$ be an arbitraly point in $\tilde{U}_g \cap \tilde{U}_{g'}$. Then there exist $\hat{x} = (g,\hat{n},\hat{t})$ and $\hat{x}' = (g',\hat{n}',\hat{t}')$ satisfying $\pi(\hat{x}) = \pi(\hat{x}') = \tilde{q}$. We denote by $P_{\Theta_{\hat{x}}}(K)_0$ the connected component of $P_{\Theta_{\hat{x}}}(K)$ containing 1. Then $P_{\Theta_{\hat{x}}}(K) = P_{\Theta_{\hat{x}}}(K)_0 M$. Since $g\hat{n}a(\hat{x})P_{\Theta_{\hat{x}}}(K) = g'\hat{n}'a(\hat{x}')P_{\Theta_{\hat{x}}}(K)$ we have $(g\hat{n}a(\hat{x}))^{-1}g'\hat{n}'a(\hat{x}') = \hat{p}\hat{m}$ with $\hat{m} \in M$ and $\hat{p} \in P_{\Theta_{\hat{x}}}(K)_0$. Since we can choose $Y'_1,\dots,Y'_{k'}$ in $\mathfrak{m}_\Theta(K) + \mathfrak{a}_\Theta + \mathfrak{n}^+_\Theta$ so that $\hat{p} = \exp Y'_{k'} \exp Y'_{k'-1} \cdots \exp Y'_1$, we see by the same argument as in the case of $\varphi^{-1}_{g\hat{n}a(\hat{x})} \circ \varphi_g$ that $\varphi^{-1}_{g\hat{n}a(\hat{x})\hat{p}} \circ \varphi_{g\hat{n}a(\hat{x})}$ defines an analytic diffeomorphism between suitable neighbourhoods of $(1, \mathrm{sgn}\,\hat{x})$. Moreover, since $\varphi^{-1}_{g\hat{n}a(\hat{x})\hat{p}\hat{m}} \circ \varphi_{g\hat{n}a(\hat{x})\hat{p}}((n,t)) = (\hat{m}^{-1}n\hat{m}, t)$, $\varphi^{-1}_{g'\hat{n}'a(\hat{x}')} \circ \varphi_{gna(\hat{x})\hat{p}}$ is an analytic diffeomorphism of $N^- \times \mathbb{R}^\ell$. Thus we have proved that $\varphi^{-1}_g \circ \varphi_{g\hat{n}a(\hat{x})}$, $\varphi^{-1}_{g'} \circ \varphi_{g'\hat{n}'a(\hat{x}')}$,

$\varphi^{-1}_{g\hat{n}a(\hat{x})\hat{p}} \circ \varphi_{g\hat{n}a(\hat{x})}$ and $\varphi^{-1}_{g'\hat{n}'a(\hat{x}')} \circ \varphi_{gna(\hat{x})\hat{p}}$ define analytic diffeomorphisms of suitable open neighbourhoods of $(1, \mathrm{sgn}\ \hat{x})$ to open subsets of $N^- \times \mathbb{R}^\ell$. Combining these maps and their inverse, we see that $\varphi^{-1}_{g'} \circ \varphi_g$ defines an analytic diffeomorphism of an open set containing $(\hat{n}, \hat{t})$ to an open set containing $(\hat{n}', \hat{t}')$, which implies $\varphi^{-1}_g(U_g \cap U_{g'})$ and $\varphi^{-1}_{g'}(U_g \cap U_{g'})$ are open in $N^- \times \mathbb{R}^\ell$ and that the map (2.12) is an analytic local diffeomorphism. But the map is bijective, so we have the claim of Lemma 7. q.e.d.

<u>Proof of Theorem 5.</u> First we remark that the proof of Lemma 7 shows that

$$(2.14) \qquad \psi_g : \pi^{-1}(\tilde{U}_g) \ni (g', n', t') \longmapsto \varphi^{-1}_g \circ \varphi_{g'}((n', t')) \in N^- \times \mathbb{R}^\ell$$

defines a real analytic map of the open subset $\pi^{-1}(\tilde{U}_g)$ of $\hat{X}$ to $N^- \times \mathbb{R}^\ell$. Therefore for any open subset V of $N^- \times \mathbb{R}^\ell$, $\pi^{-1} \circ \varphi_g(V) = \psi^{-1}_g(V)$ is open in $\hat{X}$. On the other hand, for any open subset $\hat{V}$ of $\hat{X}$, $\varphi^{-1}_g \circ \pi(V)$ is clearly open in $N^- \times \mathbb{R}^\ell$. Hence the map (2.10) is a homeomorphism.

For points x and x' in $\tilde{X}$, there exists g in G such that $\tilde{U}_g$ contains x and x' because Lemma 1 shows that $\{g \in G\ ;\ \tilde{U}_g \ni x$ and $\{g \in G\ ;\ \tilde{U}_g \ni x'\}$ are open dense in G. Since φ_g is homeomorphic and $N^- \times \mathbb{R}^\ell$ is Hausdorff, $\tilde{X}$ is also Hausdorff.

Thus we see that X is a connected real analytic manifold. The claims ii) and iii) are clear from what we have proved. The claim concerning $\tilde{X} - \tilde{U}_g$ immediately follows from Iwasawa decomposition (1.2) and Lemma 1.

Whitney's transversality theorem says that for any submanifold $\tilde{Y}$ of $\tilde{X}$ satisfying $\mathrm{codim}_{\tilde{X}} \tilde{Y} \geq 2$ and for any differentiable map $\gamma : S^1$ (= the unit circle) $\longrightarrow \tilde{X}$, there exists a differentiable map $\gamma' : S^1 \longrightarrow \tilde{X} - \tilde{Y} \subset \tilde{X}$ such that γ is homotopic to γ'.

Therefore the fundamental group of $\tilde{X}$ equals that of $\tilde{U}_g$. Since the fundamental group of $\tilde{U}_g$ is trivial, $\tilde{X}$ is simply connected.

Consider the compact subset $B = K \times \{1\} \times [-1,1]^{\ell}$ of $\hat{X}$. Then $\pi(B)$ is also compact because it is the image of a compact set under the continious map. Since $\{\exp(-\sum_{j=1}^{\ell} H_j \log t_j); 0 < t_j \leq 1$ for $1 \leq j \leq \ell\}$ equals $\overline{A_+}$, Cartan decomposition (1.1) shows that $\pi(B)$ contains all the open G-orbits of $\tilde{X}$. Therefore the compact set $\pi(B)$ is open dense in $\tilde{X}$, which implies $\pi(B) = \tilde{X}$ and that $\tilde{X}$ is compact.

To prove the claim in iv), we prepare the following:

<u>Lemma 8.</u> Let Y be an element of the Lie algebra $\mathfrak{a} + \mathfrak{n}^-$. Then by the identification

$$\begin{array}{ccc} N^-A & \xleftarrow{\sim} & N^- \times \mathbb{R}_+^{\ell} \hookrightarrow N^- \times \mathbb{R}^{\ell}, \\ \Psi & & \Psi \\ na = n\exp(-\sum_{j=1}^{\ell} H_j \log t_j) & \longleftarrow & (n,t) \end{array}$$

the left invariant vector field $Y|N^- \times \mathbb{R}_+^{\ell}$ on the Lie group N^-A corresponding to Y is expressed as

$$Y|N^- \times \mathbb{R}_+^{\ell} = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda_i} t^{\lambda} X_{-\lambda_i} - \sum_{j=1}^{\ell} c_j t_j \partial/\partial t_j,$$

where

$$Y = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda_i} X_{-\lambda_i} + \sum_{j=1}^{\ell} c_j H_j .$$

Therefore $Y|N^- \times \mathbb{R}_+^{\ell}$ can be analytically extended to a vector field on $N^- \times \mathbb{R}^{\ell}$.

<u>Proof.</u> For $a = \exp(-\sum_{j=1}^{\ell} H_j \log t_j)$, we have

$$\mathrm{Ad}(a) X_{-\lambda_i} = e^{-\lambda \log a} X_{-\lambda_i} = t^{\lambda} X_{-\lambda_i},$$

which proves the claim (cf. the proof of Lemma 7). q.e.d.

Now we will prove iv). For a Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, we denote by $U(\mathfrak{b})$ the universal enveloping algebra of $\mathfrak{b}_{\mathbb{C}}$ and naturally identify $U(\mathfrak{b})$ with a subalgebra of $U(\mathfrak{g})$. Let

$\mu : \mathbb{D}(G)^K \to \mathbb{D}(G/K)$ be the natural surjective map with the kernel $\mathbb{D}(G)^K \cap U(\mathfrak{g})\mathfrak{k}$. Then for $D \in \mathbb{D}(G)^K$, there exists a unique element $D' \in U(\mathfrak{a} + \mathfrak{n}^-)$ such that $D' \equiv D \mod U(\mathfrak{g})\mathfrak{k}$ because of the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^-$. Since $D' - D \in U(\mathfrak{g})\mathfrak{k}$, Lemma 8 proves that $\mu(D)$ can be analytically extended to a differential operator on $\tilde{U}_g$ for every $g \in G$. Therefore we have the analytic extension $\tilde{D}$ of $\mu(D)$ on $\tilde{X}$ because $\tilde{X}$ is simply connected. Let τ_g be the transformation on $\tilde{X}$ corresponding to the action of $g \in G$. Since $\tau_g^* \tilde{D} - \tilde{D}$ vanishes on the open subset G/K of $\tilde{X}$, we have $\tau_g^* \tilde{D} = \tilde{D}$ on $\tilde{X}$, which shows $\tilde{D} \in \mathbb{D}(\tilde{X})$. Hence the map $\mathbb{D}(\tilde{X}) \to \mathbb{D}(G/K)$ is surjective and the injectivity of the map is clear because G/K is open in $\tilde{X}$.

Now we remember the concept of regular singularity in [9] and the structure of $\mathbb{D}(G/K)$ (cf. Chapter X in [6]). Let $(x_1, \ldots, x_n, t_1, \ldots, t_\ell)$ be a local coordinate system of $\tilde{X}$ such that $\tilde{X}_j$ is defined by $t_j = 0$ for every $j = 1, \ldots, \ell$. Put $\vartheta_j = t_j \partial/\partial t_j$, $\vartheta = (\vartheta_1, \ldots, \vartheta_\ell)$ and $tD_x = (t_1 \partial/\partial x_1, t_1 \partial/\partial x_2, \ldots, t_\ell \partial/\partial x_n)$. Let P_j be differential operators of order r_j $(j=1, \ldots, \ell)$ on $\tilde{X}$ whose coefficients are real analytic functions. Then the system of differential equations

$$\mathcal{M} : P_j u = 0 \quad \text{for } j = 1, \ldots, \ell$$

is said to have regular singularity along the set of walls $\{\tilde{X}_1, \ldots, \tilde{X}_\ell\}$ if the following conditions hold:

[RS-0] There are differential operators $Q^i_{j,k}$ of order $< r_j + r_k - r_i$ such that

$$[P_j, P_k] = \sum_{i=1}^{\ell} Q^i_{j,k} P_i \quad \text{for } j,k = 1, \ldots, \ell.$$

[RS-1] For any j, P_j is of the form

$$P_j = P_j(t, x, \vartheta, tD_x).$$

[RS-2] Put $a_j(x,s) = P_j(0,x,s,0)$ and let $\mathring{a}_j(x,t)$

be its homogeneous part of degree r_j with respect to s. Then the solution of the system of equations

$$\mathring{a}_1(x,s) = \cdots = \mathring{a}_\ell(x,s) = 0$$

is only the origin $s = 0 \in \mathbb{C}^\ell$ for any x.

For $D \in \mathbb{D}(G)^K$, let $D'_{\mathfrak{a}}$ be a unique element of $U(\mathfrak{a})$ defined by the equation

$$D - D'_{\mathfrak{a}} \in \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k} \tag{2.15}$$

and put

$$D_{\mathfrak{a}} = e^{\rho} \circ D'_{\mathfrak{a}} \circ e^{-\rho} \tag{2.16}$$

where e^{ρ} is the function on A defined by $e^{\rho}(a) = e^{\rho \log a}$ for $a \in A$. Then denoting by $U(\mathfrak{a})^W$ the subalgebra $\{D \in U(\mathfrak{a})\ ;\ \mathrm{Ad}(m_w)D = D$ for $w \in W\}$ of $U(\mathfrak{a})$, the map

$$\begin{array}{rccc} \tilde{\gamma}\ : & \mathbb{D}(G)^K & \longrightarrow & U(\mathfrak{a}) \\ & \cup & & \cup \\ & D & \longmapsto & D_{\mathfrak{a}} \end{array}$$

defines a surjective homomorphism of $\mathbb{D}(G)^K$ onto $U(\mathfrak{a})^W$ with the kernel $\mathbb{D}(G)^K \cap U(\mathfrak{g})\mathfrak{k}$. Therefore it induces the isomorphism

$$\gamma\ :\ \mathbb{D}(G/K) \simeq \mathbb{D}(G)^K/\mathbb{D}(G)^K \cap U(\mathfrak{g})\mathfrak{k} \xrightarrow{\sim} U(\mathfrak{a})^W. \tag{2.17}$$

Here the order of $\gamma(D)$ equals that of D for $D \in \mathbb{D}(G/K)$ and $U(\mathfrak{a})^W$ is known to be a polynomial ring over $\mathbb{C}$ with ℓ algebraically independent homogeneous elements $p_1(H_1,\ldots,H_\ell),\ldots,p_\ell(H_1,\ldots,H_\ell)$.

Now we will verify the conditions [RS-0], [RS-1] and [RS-2] for the system $\mathcal{M}_\chi$, which is expressed as

$$\mathcal{M}_\chi\ :\ (D_j - \chi(D_j))\,u = 0 \quad \text{for } j = 1,\ldots,\ell,$$

where $D_j = \gamma^{-1}(p_j)$. Since $\mathbb{D}(G/K)$ is a commutative ring, [RS-0] is clear. Moreover Lemma 8 shows [RS-1] and that in [RS-2]

$$a_j(x,s_1,\ldots,s_\ell) = p_j(\rho(H_1) - s_1,\ldots,\rho(H_\ell) - s_\ell) - \chi(D_j). \tag{2.18}$$

Therefore the system of equations $\mathring{a}_j(x,s) \equiv p_j(-s) = 0$ for $j = 1,\ldots,\ell$ implies $s = 0$.

Thus we complete the proof of Theorem 5.

The following proposition will be used in a subsequent paper.

<u>Proposition 9.</u> We denote by τ_{Θ} the involutive automorphism of $\tilde{X}$ induced by the map of $\hat{X}$: $(g,n,t) \mapsto (g,n,s)$, where $s_i = t_i$ if $\alpha_i \notin \Theta$ and $s_j = -t_j$ if $\alpha_j \in \Theta$. Then τ_{Θ} and the action of G are commutative mutually and $\tau_{\Theta}^* D = D$ for any $D \in \mathbb{D}(\tilde{X})$.

<u>Proof.</u> The commutativity is clear seeing Definition 4.

For $\tilde{D} \in U(\mathfrak{g})$, we denote by $\tilde{D}'$ the unique element in $U(\mathfrak{a}+\mathfrak{n}^-)$ satisfying $\tilde{D} - \tilde{D}' \in U(\mathfrak{g})\mathfrak{k}$. This correspondence induces the identification

$$\mathbb{D}(\tilde{X}) \xrightarrow{\sim} \mathbb{D}(G/K) \hookrightarrow U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k} \xrightarrow{\sim} U(\mathfrak{a}+\mathfrak{n}^-).$$

Consider in the open submanifold $\tilde{U}_1$ of $\tilde{X}$. The totality of left N^-A-invariant differential operators on $\tilde{U}_1$ is naturally identified with $U(\mathfrak{a}+\mathfrak{n}^-)$ (cf. Lemma 8). Since $\tau_{\Theta}(\tilde{U}_1) = \tilde{U}_1$, τ_{Θ} induces an involutive automorphism τ_{Θ}^* of $U(\mathfrak{a}+\mathfrak{n}^-)$, which satisfies

$$(2.19) \qquad \begin{cases} \tau_{\Theta}^*(H_j) = H_j, \\ \tau_{\Theta}^*(X_{-\lambda_i}) = (-1)^{\sum_{\alpha_j \in \Theta} \lambda(H_j)} X_{-\lambda_i}. \end{cases}$$

Using the identifications, D in $\mathbb{D}(\tilde{X})$ can be expressed as

$$D = (D - D'_{\mathfrak{a}}) + D'_{\mathfrak{a}},$$

where $D - D'_{\mathfrak{a}} \in \mathfrak{n}^- U(\mathfrak{a}+\mathfrak{n}^-)$ and $e^{\rho} \circ D'_{\mathfrak{a}} \circ e^{-\rho} \in U(\mathfrak{a})^W$. Since $\tau_{\Theta}^* D'_{\mathfrak{a}} = D'_{\mathfrak{a}}$ and $\tau_{\Theta}^* D \in \mathbb{D}(\tilde{X})$, we have $\tau_{\Theta}^* D - D \in \mathfrak{n}^- U(\mathfrak{a}+\mathfrak{n}^-) \cap \mathbb{D}(G/K)$. Therefore the isomorphism (2.17) proves $\tau_{\Theta}^* D - D = 0$. q.e.d.

Put $\hat{X}_0 = G \times \mathbb{R}^l$ and identify $\hat{X}_0$ with the closed submanifold $G \times \{1\} \times \mathbb{R}^l$ of $\hat{X}$. Then $\hat{X}_0$ has the analytic action of G and the equivalence relation $\sim$ induced by those on $\hat{X}$. We remark here that the analytic map $\pi|X_0 : \hat{X}_0 \to \tilde{X}$, which will be denoted by

π_0, induces a homeomorphism of the quotient space $\hat{X}_0/\sim$ with the quotient topology onto $\tilde{X}$ because the map $r: \hat{X} \ni (g,n,t) \mapsto (gn,t) \in \hat{X}_0$ satisfies $r(\hat{x}) = \hat{x}$ for $\hat{x} \in \hat{X}$ (cf. Bourbaki [3]). Let $\hat{x} = (g, t)$ be a point of $\hat{X}_0$. Then by the natural identifications $T_{\hat{x}}\hat{X}_0 \simeq \mathfrak{g} + T_t\mathbb{R}^\ell$ and $T_{\pi(\hat{x})}\tilde{U}_g \simeq \mathfrak{n}^- + T_t\mathbb{R}^\ell$, the differential $(d\pi_0)_{\hat{x}}$ is expressed as

$$
\begin{aligned}
(2.20) \qquad & (d\pi_0)_{\hat{x}}(\partial/\partial t_j) = \partial/\partial t_j, \quad j = 1,\dots,\ell, \\
& (d\pi_0)_{\hat{x}}(Y) = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda_i}(g)t^{2\lambda} + c_{-\lambda_i}(g))X_{-\lambda_i} \\
& \qquad - \sum_{j=1}^{\ell} c_j(g) t_j \partial/\partial t_j, \quad Y \in \mathfrak{g},
\end{aligned}
$$

where

$$
\begin{aligned}
(2.21) \qquad \mathrm{Ad}(g^{-1})Y = & \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda_i}(g)X_{\lambda_i} + c_{-\lambda_i}(g)X_{-\lambda_i}) \\
& + \sum_{j=1}^{\ell} c_j(g)H_j \mod \mathfrak{m},
\end{aligned}
$$

(cf. (2.13)). Therefore π_0 is smooth, that is, $(d\pi_0)_{\hat{x}}$ is surjective for any $\hat{x} \in \hat{X}_0$. Moreover $\tilde{X}$ has the following universal property.

<u>Proposition 10.</u> Given an analytic map f of $\hat{X}_0$ to a real analytic manifold $\tilde{Y}$ such that $f(\hat{x}) = f(\hat{x}')$ if $\hat{x} \sim \hat{x}'$ in $\hat{X}_0$, then there is a unique analytic map $\bar{f}$ of $\tilde{X}$ to $\tilde{Y}$ such that the following diagram is commutative:

$$
\begin{tikzcd}
\hat{X}_0 \arrow[r, "f"] \arrow[d, "\pi_0"'] & \tilde{Y} \\
\tilde{X} \arrow[ur, "\bar{f}"'] &
\end{tikzcd}
$$

<u>Proof.</u> We have only to prove the analyticity of $\bar{f}$. Let s_g be the analytic map of $\tilde{U}_g$ to X_0 defined by $(n,t) \mapsto (g,n,t)$. Since $\bar{f}|\tilde{U}_g = \bar{f}\circ\pi_0\circ s_g|\tilde{U}_g = f\circ s_g|\tilde{U}_g$ for $g \in G$, $\bar{f}$ is also analytic.

q.e.d.

In [8] another realization of G/K and G/P is given. The following proposition shows the relation between the realization in [8] and $\widetilde{X}$.

Proposition 11. The natural map

$$K\times(-1,1)^{\ell} \hookrightarrow G\times\mathbb{R}^{\ell} \xrightarrow{\pi_0} \widetilde{X}$$

induces an analytic diffeomorphism

$$\iota : (K/M)\times(-1,1)^{\ell} \to \widetilde{X}$$

onto an open dense submanifold of $\widetilde{X}$ which contains G/P.

Proof. Let $\hat{x} = (k,t)$ be a point of $K\times(-1,1)^{\ell}$. Then the following

$$\begin{cases} (d\pi_0)_{\hat{x}}(\partial/\partial t_j) = \partial/\partial t_j, \quad j = 1,\dots,\ell, \\ (d\pi_0)_{\hat{x}}(Ad(k)(X_{\lambda_i} - X_{-\lambda_i})) = (t^{2\lambda}-1)X_{-\lambda_i}, \quad \lambda \in \Sigma^+,\ i = 1,\dots,m(\lambda), \end{cases}$$

shows that the map $d\pi_0 : T_{\hat{x}}(K\times(-1,1)^{\ell}) \to T_{\pi(\hat{x})}\widetilde{X}$ is surjective because $Ad(k)(X_{\lambda_i} - X_{-\lambda_i}) \in \mathfrak{k}$ and $t^{2\lambda} - 1 \neq 0$. Moreover, since $(k,t) \sim (km,t)$ for any $m \in M$, which is clear because $kma(\hat{x})P\Theta_{\hat{x}}(K) = ka(\hat{x})P\Theta_{\hat{x}}(K)$, we obtain the smooth analytic map $\iota : (K/M)\times(-1,1)^{\ell} \to \widetilde{X}$. Comparing the dimentions of the manifolds, we see that ι is analytic local diffeomorphism.

Here we note that Cartan decomposition (1.1) induces the analytic diffeomorphism

$$\begin{array}{ccc} K/M\times A^+ & \longrightarrow & G/K \\ \cup\!\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\!\shortmid \\ (kM,a) & \longmapsto & kaK \end{array}$$

onto an open dense submanifold of G/K. Therefore putting $\widetilde{Z} = K/M\times\{(-1,1)-\{0\}\}^{\ell}$, we see that the restriction $\iota|\widetilde{Z}$ is injective and $\iota(\widetilde{Z})$ is open dense in $\widetilde{X}$. Since $\widetilde{Z}$ is open dense in $(K/M)\times(-1,1)^{\ell}$ and $\iota|\widetilde{Z}$ is injective and ι is an analytic local diffeomorphism, we can conclude that ι is injective. Thus we can identify $(K/M)\times(-1,1)^\ell$ with an open dense submanifold of $\tilde{X}$. Moreover, since K acts transitively on G/P, we have $K/M\times\{0\}^\ell \xrightarrow{\sim} G/P$ by Definition 4 and Theorem 5. q.e.d.

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